Simplified Algorithm for Implementing an ABCD Ray Matrix Wave-Optics Propagator

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Outline

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  – Ray Matrices
  – Siegman Decomposition Algorithm
• Modifications to the Siegman ABCD Decomposition Algorithm
  – Simplification by Removing One Step
  – Addressing Degeneracies and Details
• Comparison of ABCD and Sequential Wave-Optics Propagation
• Conclusions
Introduction & Motivation

- Model propagation of a beam through a complex system of simple optics in as few steps as possible.
- We developed a technique for using ray matrices to include image rotation and reflection image inversion in wave-optics modeling.
- Here we introduce a technique to prescribe a wave-optics propagation using a ray matrix.
Ray Matrix Formalism
Introduction - Ray Matrices

• The most common ray matrix formalism is 2x2 – a.k.a. ABCD matrix

• It describes how a ray height, $x$, and angle, $\theta_x$, changes through a system.

\[
\begin{bmatrix}
    x' \\
    \theta'_{x}
\end{bmatrix} =
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    x \\
    \theta_x
\end{bmatrix}
\]

\[
x' = Ax + B\theta_x
\]

\[
\theta'_{x} = Cx + D\theta_x
\]
2x2 Ray Matrix Examples

**Propagation**

\[ x' = x + \theta_x L \]

\[
\begin{bmatrix}
x'
\
\theta_x'
\end{bmatrix}
=
\begin{bmatrix}
1 & L
0 & 1
\end{bmatrix}
\begin{bmatrix}
x
\theta_x
\end{bmatrix}
\]

**Lens**

\[ \theta_x' = \theta_x - x / f \]

\[
\begin{bmatrix}
x'
\
\theta_x'
\end{bmatrix}
=
\begin{bmatrix}
1 & 0
-1 / f & 1
\end{bmatrix}
\begin{bmatrix}
x
\theta_x
\end{bmatrix}
\]
## Example ABCD Matrices

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>Form</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propagation</td>
<td>$\begin{bmatrix} 1 &amp; L/n \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$L = \text{physical length}$ $n = \text{refractive index}$</td>
</tr>
<tr>
<td>Lens</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ -1/f &amp; 1 \end{bmatrix}$</td>
<td>$f = \text{effective focal length}$</td>
</tr>
<tr>
<td>Curved Mirror (normal incidence)</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ -2/R &amp; 1 \end{bmatrix}$</td>
<td>$R = \text{effective radius of curvature}$</td>
</tr>
<tr>
<td>Curved Dielectric Interface (normal incidence)</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ -(n_2-n_1)/R &amp; 1 \end{bmatrix}$</td>
<td>$n_1 = \text{starting refractive index}$ $n_2 = \text{ending refractive index}$ $R = \text{effective radius of curvature}$</td>
</tr>
</tbody>
</table>
3x3 and 4x4 Formalisms

• Siegman’s *Lasers* book describes two other formalisms: 3x3 and 4x4

• The 3x3 formalism added the capability for tilt addition and off-axis elements.

• The 4x4 formalism included two-axis operations like axis inversion and image rotation.
5x5 Formalism

- We use a 5x5 ray matrix formalism as a combination of the 2x2, 3x3, and 4x4.
  - Previously introduced by Paxton and Latham
- Allows modeling of effects not in wave-optics.
  - Image Rotation
  - Reflection Image Inversion
Ray Matrix Wave-Optics Propagation Introduction

- Introduced a way of applying effects captured by a 5x5 ray matrix model with wave-optics.
  - Image Inversion
  - Image Rotation
- This relied on a parallel sequential wave-optics model and integration of these effects at the end.
- We complete the integration technique here by showing how the residual dual-axis ABCD matrices embedded in a 5x5 ray matrix can be used to specify a wave-optics propagation.
ABCD Ray Matrix Wave-Optics Propagator
Implementation Options

- Siegman combined the ABCD terms directly in the Huygens integral.
  - Less intuitive
  - Cannot obviously be built from simple components

- He then also introduced a way of decomposing any ABCD propagation into 5 individual steps.

\[
U_2(x_2, y_2) = \frac{\exp(jkL)}{j\lambda B}.
\]

\[
\int \int U_1(x_1, y_1) \exp\left(\frac{jk}{2B} \begin{pmatrix} A(x_1^2 + y_1^2) - & B(x_1 x_2 + y_1 y_2) + \\ C(x_2^2 + y_2^2) & D \end{pmatrix}\right) \, dx_1 \, dy_1
\]

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} M_2 & 0 \\ 0 & 1/M_2 \end{bmatrix}.
\]

\[
\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & 1/M_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}.
\]
Siegman Decomposition Algorithm

• Choose magnifications $M_1$ & $M_2$ ($M = M_1 \times M_2$)

• Calculate the effective propagation length and the focal lengths.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow$$

$$L_{eq} = \frac{L}{M_1^2} = \frac{B}{M}$$

$$f_1 = \frac{B}{M - A}$$

$$f_2 = \frac{B}{1/M - D}$$
Modifications to the Siegman Decomposition Algorithm

- We found that one of the magnification terms was unnecessary (M₁=1.0).
- We modified Siegman’s algorithm to better address two important situations:
  - image planes and
  - focal planes.
- We worked on how add diffraction into choosing magnification.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1/f₂ & 1
\end{bmatrix}.
\]

\[
\begin{bmatrix}
M & 0 \\
0 & 1/M
\end{bmatrix} \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-1/f₁ & 1
\end{bmatrix}
\]

\[
D₂ = AD₁ + 2\eta \frac{L\lambda}{D₁}
\]
Eliminating a Magnification Term

- We determined that one of the two magnification terms that Siegman put into his decomposition was unnecessary.
  - There were five steps \((f_1, M_1, L, M_2, f_2)\) and four inputs \((ABCD)\).

Original Decomposition

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \\
\begin{bmatrix}
1 & 0 \\
-1/f_2 & 1
\end{bmatrix} \begin{bmatrix}
M_2 & 0 \\
0 & 1/M_2
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1/M_1
\end{bmatrix} \begin{bmatrix}
-1/f_1 & 1
\end{bmatrix}
\]

New Decomposition

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \\
\begin{bmatrix}
1 & 0 \\
-1/f_2 & 1
\end{bmatrix} \begin{bmatrix}
M & 0 \\
0 & 1/M
\end{bmatrix} \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix} \begin{bmatrix}
-1/f_1 & 1
\end{bmatrix}
\]
Image Plane: B=0

- This case is an image plane.
- There is no propagation involved here, but there is
  - curvature and
  - magnification.

\[
\begin{bmatrix}
1 & 0 \\
-1/ f_2 & 1
\end{bmatrix}
\begin{bmatrix}
M & 0 \\
0 & 1/ M
\end{bmatrix}
= 
\begin{bmatrix}
M & 0 \\
-M / f_2 & 1/ M
\end{bmatrix}
\]

**Siegman**

\[
L_{eq} = \frac{B}{M} = 0
\]

\[
f_1 = \frac{B}{M - A} = 0
\]

\[
f_2 = \frac{B}{1/ M - D} = 0
\]

**Our Algorithm**

\[
L_{eq} = 0
\]

\[
C = -1/Mf_2
\]

\[
f_2 = \frac{-1}{MC}
\]
Automated Magnification Determination: Problems with the Focal Plane

- We were trying to automate the selection of the magnification by setting it equal to the A term of the ABCD matrix.
  - This minimizes the mesh requirements
- In doing so, we found that the decomposition algorithm was problematic at a focal plane.

\[
\begin{bmatrix}
1 & f \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1/f & 1
\end{bmatrix}
= \begin{bmatrix}
0 & f \\
-1/f & 1
\end{bmatrix}
\]

Siegman, M=A

\[
M = A = 0 \rightarrow
\]

\[
L_{eq} = \frac{f}{M} = \infty
\]

\[
f_1 = \frac{f}{M - A} = \frac{f}{0}
\]

\[
f_2 = \frac{f}{\infty - 1} = 0
\]
Propagation to a Focus: $A=0$

$$
\begin{bmatrix}
1 & f \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
-1/f
\end{bmatrix}
= 
\begin{bmatrix}
0 & f \\
-1/f & 1
\end{bmatrix}
$$

- For a collimated beam going to a focus, this ray envelope diameter is zero.
- To handle this case, we force the user to specify the magnification.
- We also give the user guidance on how to choose magnification when there is substantial diffraction...

**Siegman, $M=A$**

\[
M = A = 0 \rightarrow
\]

\[
L_{eq} = \frac{f}{M} = \infty
\]

\[
f_1 = \frac{f}{M - A} = \frac{f}{0} = \infty
\]

\[
f_2 = \frac{f}{\infty - 1} = 0
\]

**Siegman, $M=1$**

\[
M = 1 \rightarrow
\]

\[
L_{eq} = \frac{B}{M} = f
\]

\[
f_1 = \frac{B}{M - A} = f
\]

\[
f_2 = \frac{B}{1/M - D} = 0
\]
Choosing Magnification while Considering Diffraction

- We propose here to add a diffraction term to the magnification to avoid the case of small $M$.
- We added a tuning parameter, $\eta$, which is the number of effective diffraction limited diameters.

\[
D_2 = AD_1 + 2\eta \frac{L\lambda}{D_1}
\]

\[
M = \frac{D_2}{D_1} = A + 2\eta \frac{L\lambda}{D_1^2}
\]

\[
= A + \frac{\eta}{2} \frac{1}{N_f}
\]
Common Diffraction Patterns

- Airy
- Sinc
- Gaussian

Normalized Intensity vs. Normalized Radius
We concluded that $\eta=5$ is sufficient to capture more than 99% of the 1D integrated energy.
Modified Decomposition Algorithm

• If at an image plane (B=0)
  - M=A (possible need for interpolation)
  - Apply focus

• Else
  - Specify M, considering diffraction if necessary
  - Calculate and apply the effective propagation length and the focal lengths.

\[
\begin{bmatrix}
M - \frac{LM}{f_1} & LM \\
\frac{LM}{f_1 f_2} - \frac{1}{M f_1} - \frac{M}{f_2} & 1 - \frac{LM}{M f_2}
\end{bmatrix}
\]

for \( M_1 = 1.0 \)

\[
L_{eq} = \frac{L}{M_1^2} = \frac{B}{M}
\]

\[
f_1 = \frac{B}{M - A}
\]

\[
f_2 = \frac{B}{1 / M - D}
\]
Wave-Optics Implementation Details
Implementing Negative Magnification

• After going through a focus, the magnification is negated.

• We implement negative magnification by inverting the field in one or both axes.
  – We consider the dual axis ray matrix propagation using the 5x5 ray matrix formalism.
Dual Axis Implementation

- Cylindrical telescopes along the axes are handled by dividing the convolution kernel into separate parts for the two axes.

\[ U_2 = P \cdot F^{-1}(H \cdot F(U_1)) \]

\[ H = \exp[-j \pi \lambda (z_x f_x^2 + z_y f_y^2)] \]

\[ F(x) = \text{Fourier Transform of } x \]

\[ P = \text{Phase Factor} \]
WaveTrain Implementation

WaveTrain incident

Cyl Lens

F1x

Cyl Lens

Fy

Cyl Lens

vacuumpropcyl

Cyl Lens

F2x

Cyl Lens

F2y

WaveTrain transmitted

tilt

tilt

dvtorv

f $\leftarrow [V]\,$

ftovf

srmdcomp

srmcompose

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Example: ABCD Propagator
• Compared sequential and ABCD propagation fields
Field before the Lens

Field Magnitude

Field Phase
Field After Lens by Distance f/2

Field Magnitude

Field Phase
Field After Lens by Distance $f$

Field Magnitude

Field Phase
Field After Lens by Distance 3f/2

Field Magnitude

Field Phase
Field after Lens by Distance 2f

Field Magnitude

Field Phase
ABCD Ray Matrix Fourier Propagation

Conclusions

• We have modified Siegman’s ABCD decomposition algorithm to
  – remove one of the magnifications and
  – include several special cases such as
    • Image planes
    • Propagation to a focus

• This enables complex systems comprised of simple optical elements to be modeled in 4 steps (one Fourier propagation).
Questions?

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