Algorithm for Implementing an ABCD Ray Matrix Wave-Optics Propagator

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ABSTRACT

In prior work we described a 5x5 ray matrix formalism and how to integrate the effects that are not modeled in wave-optics with the ray matrix model. In this paper we describe how to complete the integration of the two techniques by modifying the Siegman ABCD ray matrix decomposition. After removing the separable effects like image rotation and image inversion, we break the 5x5 ray matrix into two 2x2 sections (a.k.a. the ABCD matrices) that correspond to the two axes orthogonal to the propagation. We then present a general algorithm that breaks any arbitrary ABCD matrix into four simple wave-optics steps. The algorithm presented has sufficient generality to handle image planes and focal planes. This technique allows for rapid and accurate wave-optics modeling of the propagation of light through complex optical systems comprised of simple optics.

Keywords: Wave-optics, Fourier optics, ABCD matrices, ray matrices, optical system modeling.

1. INTRODUCTION

Wave-optics and ray matrices are two techniques that have been used for many years to model complex optical systems. The ray matrix technique is a fast way to gain an understanding of how a ray of light propagates through a series of optics.[1] In the traditional 2x2 (a.k.a. ABCD) form, they can be used to determine whether a system is imaging, if it adds curvature to a beam, or whether it adds magnification. If more detailed system performance is required, wave-optics techniques are typically employed. Wave-optics allows for modeling of higher-order effects on the wavefront and diffraction in a way that is not possible in ray matrices, but at a significant computational cost. Wave-optics modeling of complex systems of simple optics is therefore very time consuming. This paper presents a method of reusing simple ray matrix results to predict the performance of complex optical systems using.

2. BACKGROUND

2.1. Wave Optics Modeling

Wave-optics (a.k.a. Fourier optics) models simulate the propagation of light using the Fresnel approximation of the Huygens-Fresnel principle.[2] Light propagated between two planes can be modeled using a two-dimensional Fourier transform of a grid of samples of the complex electric field. Although this technique is much more computationally intensive, it allows much more detail about the beam to be modeled including the effects of higher-order aberrations and the resulting system transfer functions like the modulation transfer function (MTF) or the optical transfer function (OTF).

Using the Huygens-Fresnel principle, the field amplitude, U, at a point in the output plane (x_2,y_2) is the superposition integral of points in the input plane, (x_1,y_1), inside some limiting aperture Σ and the Green's function of free space (a.k.a. the propagation kernel), $h(x_1,y_1;x_2,y_2)$, which is written as,

$$U(x_2,y_2)=\iint_{\Sigma} h(x_2,y_2;x_1,y_1)U(x_1,y_1)dx_1dy_1$$

where, in the case of relatively small angles,

$$h(x0,y0;x1,y1)=\frac{1}{j\cdot\lambda\cdot z}\exp\left(j\cdot k\cdot r_{12}\right),$$

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$$r_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ ,$$

and z is the axial distance between the input and output planes, λ is the wavelength of light being modeled, and k is the wave number (2π/λ). Using the Fresnel approximation, the kernel reduces to

$$h(x_1, y_1; x_2, y_2) = \frac{\exp(j \cdot k \cdot z)}{j \cdot \lambda \cdot z} \exp\left(j \frac{k}{2z} \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right] \right).$$

Using this form of the kernel, the propagation is a convolution of the input field with the kernel, and is referred to as the convolution propagator.

Further simplification allows the kernel to be decomposed into three separate terms which are two quadratic (or parabolic) phase terms in the input and output planes and a term that is directly analogous to the 2D Fourier transform operator. This form, referred to as the one-step propagator, is written as

$$U(x_2, y_2) = \frac{\exp(j \cdot k \cdot z)}{j \cdot k \cdot z} \exp\left(j \frac{kr_2^{\ 2}}{2z} \right) \iint_{-\infty}^{\infty} \left\{ U(x_1, y_1) \exp\left(j \frac{kr_1^{\ 2}}{2z} \right) \right\} \exp(-jkz(x_1 x_2 + y_1 y_2)) dx_1 dy_1$$

where

$$r_2 = \sqrt{x_2^{\ 2} + y_2^{\ 2}} \quad \text{and} \quad r_1 = \sqrt{x_1^{\ 2} + y_1^{\ 2}} \ .$$

This can be simplified into an operator notation as

$$U_2 = \frac{\exp(jkz)}{jkz} Q_2(z) \cdot F(U_1 \cdot Q_1(z))$$

where the F(…) operator is the 2D Fourier transform operation and the Q(z) is the quadratic phase factor with a radius of curvature of z, which can be written as

$$Q_x(z) = \exp\left(j \frac{kr_x^{\ 2}}{2z} \right)$$

where the subscript, x, represents the plane of the operation.

2.2. Numerical Modeling of the Convolution Propagator

The most basic wave-optics modeling can be done with a one-step Fourier propagator. Implementation of this form of propagator involves multiplying the field by the quadratic phase factor, Fourier transforming the field, and then multiplying by an additional quadratic phase factor and a scaling term. This is the fastest way of modeling optical propagation, but it has some practical disadvantages when implementing complex wave-optics models. In computational wave-optics modeling, the field is represented by samples of the complex field on a discrete 2D mesh. The spacing between mesh points needs to be chosen carefully to ensure that the quadratic phase factors and the field are properly represented.[3] For quadratic phase factors, the mesh spacing must be sufficient to represent the slope of the quadratic phase factor at the edge of the region of interest. If the region of interest has a diameter D_1, the minimal mesh spacing is

$$\frac{\lambda}{2\delta_1} \geq \frac{D_1}{2R} \quad \Rightarrow \quad \delta_1 \leq \frac{\lambda R}{D_1}$$

where δ_1 is the mesh spacing in the input field and R is the radius of curvature of the quadratic phase factor being applied in the input plane. In a one-step propagator, the output spacing between mesh points, δ_2 is $\lambda z / \delta_1$. The same sampling constraints need to be maintained for the quadratic phase factor in the output plane as well. (For more information on mesh spacing requirements, see reference 3).

Because of the lack of control over the final plane mesh spacing in the one-step Fourier propagator, a convolution propagator is often used for wave-optics simulations. The convolution propagator is typically implemented on a computer by Fourier transforming the input field, multiplying by a convolution kernel, and inverse Fourier transforming the field. Implementation of this type of propagator is very convenient because unlike the one-step Fourier propagator, the input and output results are represented with the same grid spacing.

Using a convolution propagator allows for mesh spacing to be maintained, but if the light is converging or diverging the mesh needs to be adjusted to match the beam size in order to maintain the resolution. As an example, imagine light being modeled propagating through a lens and to the focus. In most cases, if the grid size were maintained, only a few mesh points would have any light. Control over the final propagation mesh size is achieved by propagating relative to a spherical reference wave. In spherical wave propagation (SWP), the effective propagation distance and mesh size change relative to the radius of curvature of the reference . Physically, SWP is analogous to a nominally collimated beam propagating through a lens to a distance z as is shown in Figure 1. The effective propagation distance in collimated space can be calculated by the imaging equation[4]

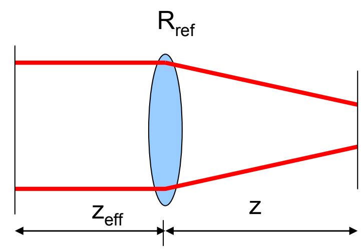

Figure 1 - Effective optical diagram for spherical wave propagation

$$\frac{1}{R_{ref}} = \frac{1}{z} + \frac{1}{z_{eff}}$$

where f is the focal length of the lens, z is the propagation distance, and z_{eff} is the effective propagation distance. Solving for z_{eff} gives,

$$z_{eff} = \frac{z - R_{ref}}{z \cdot R_{ref}}.$$

The only difference between the complex field at a plane a distance z in front of the lens with focal length f and field at the image plane z_{eff} from the lens is an effective magnification, M_{eff}, which can be calculated by

$$M_{eff} = \frac{D(z_{eff})}{D(z)} = \frac{z}{z_{eff}} = \frac{z - R_{ref}}{R_{ref}}.$$

After the propagation of distance L_{eff} and magnification of the field by M_{eff}, the output reference radius of curvature needs to be reduced by the propagation distance L_{eff}, or $R_{ref}' = R_{ref} - L_{eff}$.

As an example of a case where SWP is typically used, consider a nominally collimated beam that we want to model being transmitted through a lens with focal length f and propagating a distance L. Further assume that the regions of interest (ROI) in the input and output planes are of diameter D_1 and D_2 respectively. The magnification for this propagation is given by the ratio of the ROI diameters, so the required reference radius of curvature, R_{ref}, is given by

$$R_{ref} = z\left(\frac{D_1}{D_1 - D_2}\right).$$

The reference radius of curvature does not necessarily need to be equal to the lens focal length f. In fact, in most cases the output diameter, D2, will be larger than that predicted by simple ray tracing to account for diffractive spreading. (There is more on this topic later in this paper). The difference between the desired applied curvature, f, and the curvature in the reference needs to be applied directly to the mesh. The radius of curvature applied to the mesh is given by

$$R_{mesh}^{-1} = f^{-1} - R_{ref}^{-1}.$$

In summary, the SWP technique propagates a beam relative to a large curvature by finding an effective propagation length and magnifying the beam to compensate for the relative curvature.

2.3. Ray Matrix Modeling

Ray matrices are a common technique for modeling simple optical systems. Using the traditional ray matrix technique, a ray of light is represented by its distance from the optical axis and its angle relative to the optical axis. This technique is limited to modeling a set of ideal lenses with no higher order distortion and no apertures that significantly affect the beam.

The simplest ray matrix formalism uses a 2x2 matrix, often referred to as an ABCD matrix, to represent each optical operation on a ray of light. This formalism is sufficient for modeling the behavior of simple optical systems, but lacks the ability to handle effects like multi-dimensionality, image inversion, image rotation, image translation, and tilt addition.

There are formalisms that exist which expand the simple 2x2 matrices to handle some of these effects. Gerrard and Burch present a 3x3 matrix formalism that has been expanded to handle image translation and tilt addition.[5] A 4x4 matrix formalism is presented by Siegman that allows orthogonal axes to be modeled so that things like simple astigmatism and image rotation can be modeled.[1] In prior work, we presented an integration of these two formalisms into a single 5x5 ray matrix formalism that enables modeling of effects like tilt addition, image translation, and image rotation.[6] We also developed a technique for decomposing a given system described by a 5x5 ray matrix into a set of six effects beyond the simple wave-optics model. These effects were tilt, offset, focal power, magnification, image inversion, and image rotation. At the end of this work we referred to the possibility of removing tilt, offset, image inversion, and image rotation and being left with a simple 2x2 ray matrix (an ABCD matrix), which could be implemented with the Huygen's Propagation integral. There are numerous possible numerical methods for implementing this type of integral, but we present here an efficient algorithm that leverages fast Fourier transform (FFT) numerical techniques.

2.4. Combining the ABCD and Huygens Propagation Integral

Several authors have described techniques for integrating the Huygens propagation integral with ABCD matrices. Siegman and Herzig both describe the Huygens propagation integral in terms of the 2x2 element ray matrix (a.k.a the ABCD matrix) results of an optical system as long as no non-linear elements, like deformable mirrors or apertures are introduced as[7,8]

$$U_2(x_2, y_2) = \frac{\exp(jkL)}{j\lambda B} \int \int_{-\infty}^{\infty} U_1(x_1, y_1) \exp\left[\frac{jk}{2B}\left(A\left(x_1^{\ 2} + y_1^{\ 2}\right) - 2\left(x_1 x_2 + y_1 y_2\right) + D\left(x_2^{\ 2} + y_2^{\ 2}\right)\right)\right] dx_1 dy_1$$

where U_2 is the output electric field, U_1 is the input electric field, L is the on-axis optical path length through the system computed as the sum of each path length times its refractive index, A, B, C, and D are the four elements of the 2x2 ray matrix, λ is the wavelength, and k is the wave number ($2\pi / \lambda$). This formalism is presented here to handle only one ABCD matrix, but can be extended to two to handle differing effects in the two axes.[8] Siegman goes on in his book to decompose a general ABCD matrix into five consecutive operations: focus, magnification, propagation, magnification, and focus, or

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} M_2 & 0 \\ 0 & 1/M_2 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & 1/M_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

where f_1 and f_2 are the focal lengths, M_1 and M_2 are the magnifications, and L is the propagation distance. Siegman goes on to give some limited guidance on the choice of M_1 and M_2, but does not provide the user specific information on how to choose these parameters.

Ozaktas describes a technique for implementing wave propagation through a system with an ABCD matrix that leverages fractional Fourier transforms.[9] Ozaktas's algorithm uses scaling parameters for the input and output spaces that are not well defined. Furthermore, the Ozaktas algorithm requires that the user create a fractional Fourier transform routine for propagation which does not fit well into our traditional wave-optics formalism.

Each of these two approaches presents implementation challenges, but we chose to pursue further investigation of the Siegman algorithm because it offered the most overlap with the existing wave-optics infrastructure in our WaveTrain software.

3. REFINING SIEGMAN'S ABCD DECOMPOSITION ALGORITHM

Siegman's algorithm for decomposing an ABCD matrix produces the following relationships

$$L_{eq} = \frac{L}{{M_1}^2} = \frac{B}{M}, \qquad f_1 = \frac{B}{M-A}, \qquad f_2 = \frac{B}{1/M - D}$$

where M is the product of M_1 and M_2 and the unused element C is determined based on the constraint of the ABCD matrices that the determinant (AD-BC) is equal to 1. The algorithm description also indicates that the Fresnel number, and hence the computational difficulty, is independent of the distribution of the magnifications M_1 and M_2, but gives very little guidance on appropriate choices of magnification. Therefore, in implementing the Siegman algorithm, we first investigated the choice of the magnifications M_1 and M_2.

3.1. Choosing Magnification

Since the distributions of M_1 and M_2 did not affect computational complexity, we chose to make M_1=1.0 and have the system magnification, M, set to M_2. This reduces the number of degrees of freedom of the problem, but also the number of steps required in the modeling.

There are several logical choices for the magnification. First, in the case where the user wants to maintain the wave-optics mesh spacing, the system magnification, M, can be set to 1.0. We have found that this case is quite effective except when the system being described has an effective negative magnification. This effect can come from going through a focus or image rotation. In the case of negative system magnification, it is better to choose a magnification of -1.0.

The next logical choice of magnification is to match the system magnification. This is the A term of the ABCD matrix in the case of an imaging optical system, but can be a different value if the system is more complicated. Usually the user knows the system's magnification and can input that value directly. Making the magnification equal to the A term of the ABCD is only especially problematic when A is zero, like it is at a focus. In this case, the user must consider diffraction. To add diffraction, we defined the diameter of the output region of interest (ROI) as

$$D_2 = AD_1 + 2\eta \frac{L\lambda}{D_1}$$

where D_1 is the input beam diameter that is specified by the user, A is the term of the 2x2 ray matrix in the first row and first column, z is the propagation distance, λ is the optical wavelength, and η is a scale factor related to the amount of energy that the user wants to capture. To determine the energy captured by different values of η, the energy with respect to radius was determined for the far-field diffraction patterns of a circular aperture (an Airy pattern), a square aperture (sinc), and a Gaussian ($I=\exp[-2\,(r/w)^2]$) without a hard edge aperture but with w = D/3. Figure 2 shows the energy captured by varying values of η for these common far-field patterns. Based on this, we determined that a diffraction scale factor of 5 or a total coefficient of 10 (with the factor of 2) would be sufficient.

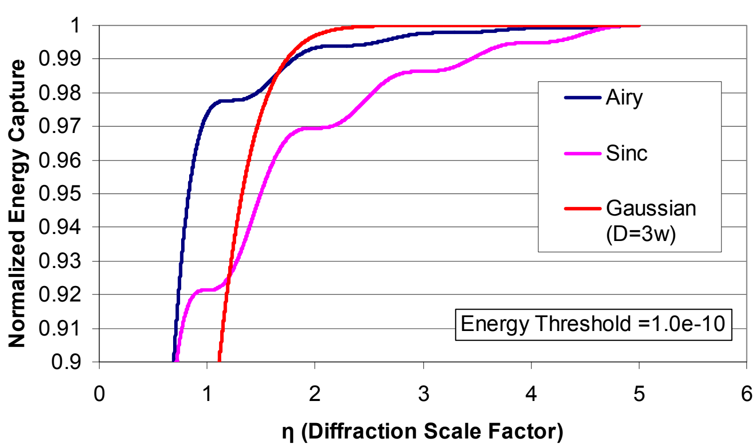

Figure 2 - Energy capture comparison of diffraction scaling factors

Based on the formula for the output diameter that included diffraction and the relationships defined by Siegman, we found that the magnification for a given system should be given by

$$M = \frac{D_2}{D_1} = \frac{A}{2} \pm \frac{\sqrt{{D_1}^2 A^2 + 4\eta\lambda B}}{2D_1}$$

where A and B are elements of the 2x2 ray matrix and D_1 is the user-specified diameter of the region of interest. We use the sign of the ray magnification term (A) to determine which solution to use. If the A term was negative, the difference between the two terms is used. If it is positive, the sum is used.

After running several cases, we found that the best choice for the system magnification in most cases is to set it equal to the expected magnification of the optical system. When doing wave-optics, it is important to maintain the sign of the magnification so that image inversion in the wave-optics field can be implemented correctly. In cases where light is being propagated to a focus, the diffraction must be considered when choosing the magnification. Although this logic could be implemented as a function, we chose to leave the choice of magnification to the user.

3.2. ABCD Matrix Decomposition Enabling Spherical Reference Wave Propagation

The advantage of propagation relative to a reference curvature is that the curvature does not have to be applied to the mesh, so the mesh spacing does not have to decrease to avoid a Nyquist sampling violation in the phase of the wave-optics field. One way of understanding the parameters of the ABCD matrix propagation is to compare them with spherical reference wave propagation. Consider a system composed of a lens of focal length f followed by a propagation of a distance d. The resulting ABCD matrix for this system is given by,

$$\begin{bmatrix} 1-d/f & d \\ -1/f & 1 \end{bmatrix}.$$

If we apply the Siegman decomposition with the magnification M equal to 1.0, the system decomposes into f_1=f, L_{eq}=d, and $f_2=\infty$, which is exactly the steps used to create the ABCD matrix.

If we set the magnification equal to the ray diameter at a distance d, the magnification becomes 1-d/f, which is exactly equal to the A term. This clearly does not work when d=f since the magnification becomes zero, but we will neglect this degenerate case since we addressed it earlier. With M=1-d/f=A, the system decomposes into $f_1=\infty$, L_{eq}=fd/(f-d), and f_2=f-d, which is mathematically identical to the decomposition derived using spherical reference wave propagation presented earlier. Further study can show that the Siegman decomposition is mathematically identical to spherical reference wave propagation and can be used to bridge the gap between ray matrix methods and wave-optics.

3.3. ABCD Propagation Implementation

The ray-matrix propagator was implemented in WaveTrain using the system shown in Figure 3. The details of the optical setup are entered as a vector into a component called SRMCompose, which creates a ray matrix (SRM is an acronym for System Ray Matrix). The ray matrix is then decomposed into steps and information for each of these steps is distributed to individual components that operate on the wave. The first step is to apply tilt to the wave. Then the wave is translated as necessary. The next steps implement the Siegman-type ABCD propagator described above. First a focus is applied separately in the two axes. Then a component called ResetCWPReference adds the reference curvature calculated in the ray matrix decomposition to the wave's reference curvature, which is held in a register separate from the wave-optics mesh. Then the wave is propagated a distance relative to its reference curvature and magnified. The propagation implemented here is done relative to separate cylindrical curvatures in the two axes. The magnification is a user-specified parameter for this system that is separate in the two axes. Magnifying a wave-optics field involves changing the mesh spacing, reducing the field amplitude by $\sqrt{M}$, reducing the tilt on the field by M, and reducing the reference curvature register by M^2. Finally the additional curvature indicated by f_2 is added separately in the two axes. In this implementation of the component, we do not handle image rotation, but that can be implemented with an interpolation step.

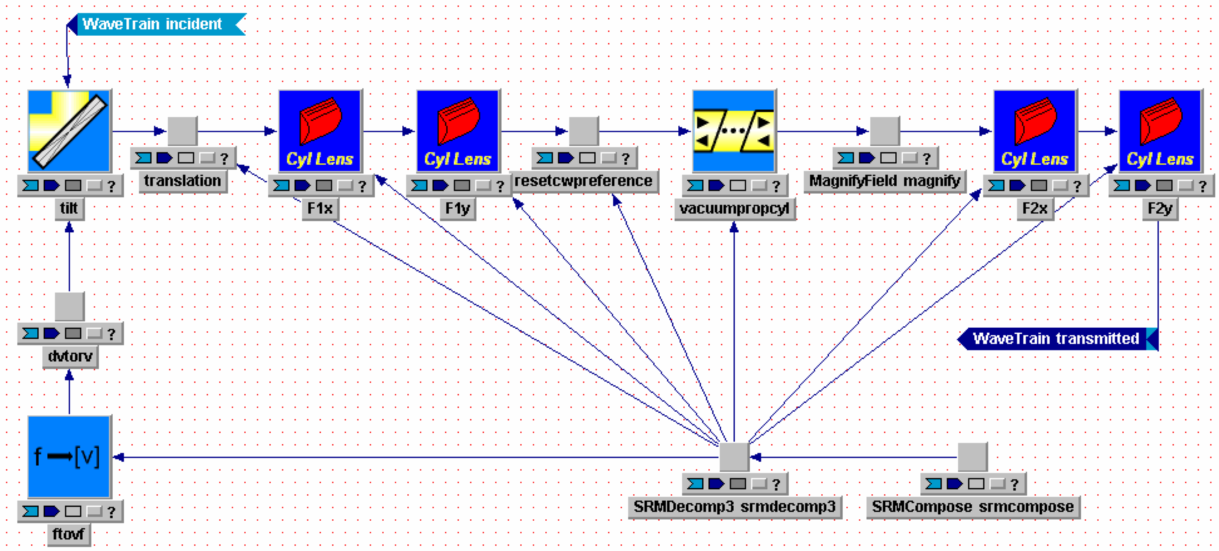

Figure 3 - WaveTrain implementation of the ABCD propagator

3.4. No Effective Propagation Exception

The only significant exception that we found to the procedure outlined by Siegman was the case of a simple lens. A simple lens has an ABCD ray matrix of

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix},$$

where f is the focal length of the lens. Using the Siegman decomposition, L_{eff}, f_1, and f_2 are zero. This degenerate case is also true when the system has magnification, which results in an ABCD matrix given by

$$\begin{bmatrix} M & 0 \\ -1/Mf & 1/M \end{bmatrix}$$

This case is particularly unusual because there is no propagation involved, but needs to be handled in order to properly model imaging systems. To handle these exceptions, we checked for the case where B=0 but C is not 0 or A is not 1. If A was not 1, we forced the output magnification to be A. In the case where C was non-zero, we set f_2 to -A/C and f_1 to zero. Another equivalent way of handling the curvature is to set f_1 to -1/CA and f_2 to zero.

4. EXAMPLE 4-F IMAGING SYSTEM

As an example, we modeled the 4-f imaging system with a hard-edge input aperture shown in Figure 4. The beam diameter, D, was 1 m. The wavelength, λ, was 1.0 µm. The lens focal length was 60 km. The beam was propagated from the aperture to the lens, and to planes 30 km apart afterward the lens to the image plane. The modeling was done both with traditional sequential convolution non-spherical wave propagation and with ABCD matrices to the desired plane and then propagated using the ABCD propagation algorithm described above.

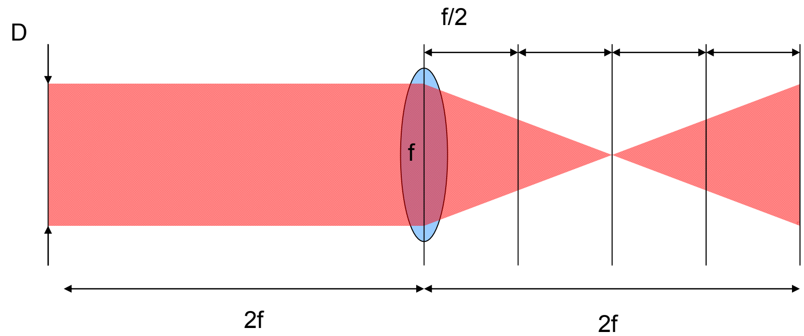

Figure 4 - Example 4-f Imaging System

Table 1 shows the ray matrices for each of the planes in symbolic form.

Table 1 – 2x2 Ray Matrices for the Example 4f Imaging System

Position	**ABCD Ray Matrix**
2f before lens	$\begin{bmatrix} 1 & 2f \\ 0 & 1 \end{bmatrix}$
2f after lens	$\begin{bmatrix} 1 & 2f \\ -1/f & -1 \end{bmatrix}$
2.5f	$\begin{bmatrix} 0.5 & 1.5f \\ -1/f & -1 \end{bmatrix}$
3f	$\begin{bmatrix} 0 & f \\ -1/f & -1 \end{bmatrix}$
3.5f	$\begin{bmatrix} -0.5 & f/2 \\ -1/f & -1 \end{bmatrix}$
4f	$\begin{bmatrix} -1 & 0 \\ -1/f & -1 \end{bmatrix}$

Figure 5 shows amplitudes of the fields in 1D from the sequential traditional wave-optics simulation and the ABCD propagation to the same plane. Figure 6 shows the unwrapped phase profiles for each of the modeled planes. To ensure proper sampling, 8192 mesh points were used in this simulation with a mesh spacing of 1.875 mm.

Most of the amplitude and phase patterns generated by the sequential propagator and ABCD propagator overlie exactly, but there are two planes with some noticeable differences. The phase in the focal plane of the imaging lens has 2π phase discontinuities. This is due to a numerical ambiguity in representing the sinc lobes of the far-field field pattern. There is no effective difference between the $-\pi$ and $+\pi$, so the computer has the option of choosing either and does so differently for the two models. The other observable difference is in the amplitude of the image plane. There is some enhanced ringing at the edges of the image of the aperture due to numerical precision of the simulation.

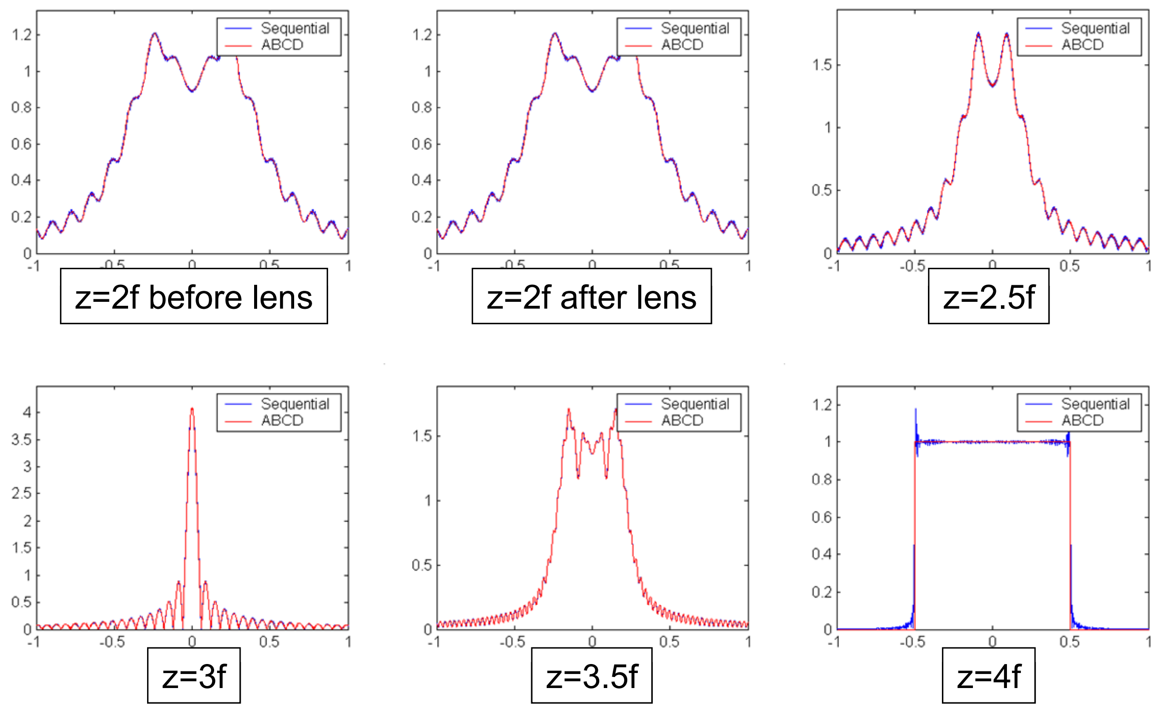

Figure 5 - Field amplitudes for the modeled planes of the 4-f imaging system

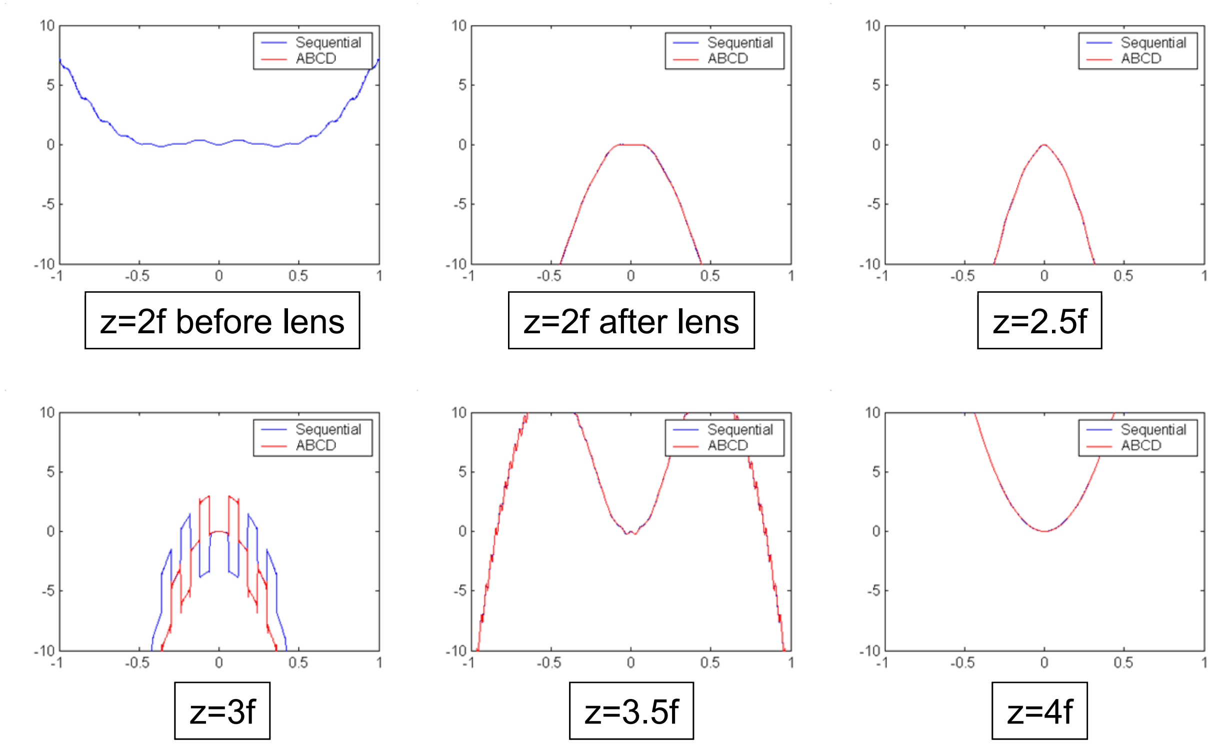

Figure 6 - Unwrapped 1D phase profiles (in radians) of the target planes of the 4-f imaging system

5. CONCLUSIONS

We present here an algorithm for propagating a beam through a complex system of simple optics using the system's ray matrix or ABCD description. This algorithm is simpler than the algorithm presented by Siegman and addresses some of degenerate cases and exceptions that the Siegman algorithm does not. Our algorithm is directly and obviously compatible with existing wave-optics codes.

Acknowledgements

This work was funded under contract to the ABL SPO. We offer our thanks to Dr. Salvatore Cusumono and Captain Jason Tellez for their continued contributions and support.

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