Technical Note

Turbulence/Optical Quick-Reference Table

Quick Reference Table of Turbulence and Optical Relations and Equivalents

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1 General Structure of the Table

A quick-reference table for turbulence and optical relations frequently used by MZA is shown in Table 1. The table consists of 8 columns numbered as shown. Relations for turbulence propagation are shown on the left half of the table (columns 1-4.) Relations for static optical aberrations are shown on the right half of the table (columns 5-8.) The Strehl ratio values for column 4 and column 5 are shared, and highlighted in bold type. These values are associated with the higher-order Strehl ratio S_h for the turbulence relations, and with a general Strehl ratio S for the static optical aberration equivalents.

Table 1: Quick-reference table of turbulence and optical relations and equivalents.

1	2	3	4	5	6	7	8
turbulence				static			
D/r_0	S	σ_T (λ/D)	S_h	S	σ_J (λ/D)	σ_{WFE} (λ)	BQ
0.48	0.75	0.23	**0.95**		0.10	0.04	1.03
0.75	0.58	0.33	**0.90**		0.15	0.05	1.05
1.18	0.37	0.49	**0.80**		0.23	0.08	1.12
1.57	0.25	0.62	**0.70**		0.29	0.10	1.20
1.96	0.18	0.75	**0.60**		0.37	0.11	1.29
2.38	0.13	0.88	**0.50**		0.45	0.13	1.41
2.84	0.10	1.02	**0.40**		0.55	0.15	1.58
3.42	0.07	1.19	**0.30**		0.69	0.17	1.83
4.19	0.05	1.41	**0.20**		0.90	0.20	2.24
5.55	0.03	1.78	**0.10**		1.35	0.24	3.16

2 Turbulence Relations

2.1 (1) D/r_0, Relative Aperture

Column 1 lists the relative aperture, i.e. the ratio of the aperture diameter D to the atmospheric coherence diameter r_0 [1]. For imaging or beam projection to a target at a finite range it is appropriate to calculate the spherical-wave coherence diameter [2] as

$$r_0 = \left[0.423 k_0^2 \int_0^L C_n^2(h(z))(1 - z/L)^{5/3} dz\right]^{-3/5}, \tag{1}$$

where $C_n^2(h(z))$ is the refractive-index structure function coefficient at the beam altitude $h(z)$, which is a function of the position z along the path, $k_0 = 2\pi/\lambda$ where λ is the wavelength of the laser, L is the slant range, and the integral extends from the platform to the target. For targets at long range such as stellar objects, it is appropriate to calculate the plane-wave r_0 as

$$r_0 = \left[0.423 k_0^2 \int_0^L C_n^2(h(z)) dz\right]^{-3/5}, \tag{2}$$

where the integral extends over the portion of the path of length L where turbulence is present.

2.2 (2) S, Tilt-Included Strehl Ratio

Column 2 lists the tilt-included Strehl ratio S for a given turbulence condition, representing the combined effect of wavefront tilt and higher-order aberrations on the on-axis intensity for an optical system. The tilt-included Strehl ratio is computed given D/r_0 by direct numerical integration of the turbulence-degraded optical transfer function relative to diffraction limit [3]

$$S = \frac{\int d\vec{f}\, \mathcal{H}_o(\vec{f})\mathcal{H}_{LE}(\vec{f})}{\int d\vec{f}\, \mathcal{H}_o(\vec{f})}, \tag{3}$$

where the integral is over all spatial frequencies $\vec{f}$ and the transfer function of the diffraction-limited optical system is given by

$$\mathcal{H}_o(\vec{f}) = \frac{W(\vec{f}\lambda d_i) \star W(\vec{f}\lambda d_i)}{W(0) \star W(0)}. \tag{4}$$

In Eq. (4) $W(\vec{f}\lambda d_i)$ is the pupil function of the aperture (ones inside pupil, zeros elsewhere,) d_i is the distance between the pupil and image plane, and where $\star$ represents autocorrelation defined by

$$f(\vec{x}) \star g(\vec{x}) = \int d\vec{x}' f(\vec{x}' - \vec{x}) g^*(\vec{x}'). \tag{5}$$

For the tilt-included Strehl ratio, the atmospheric transfer function is computed as

$$\mathcal{H}_{LE}(\vec{f}) = \exp\left\{-\frac{1}{2} 6.88 \left(\frac{\bar{\lambda} d_i |\vec{f}|}{r_o}\right)^{5/3}\right\}. \tag{6}$$

Note that since the cut-off frequency of the diffraction-limited OTF is proportional to D, the governing parameter for turbulence degradation will be D/r_0. Figure 1 shows the tilt-included Strehl ratio S as a function of D/r_0. The values indicated in Table 1 are noted on the plot. These have been set by selecting arbitrary values of the tilt-removed Strehl ratio as discussed in Sec. 2.4.

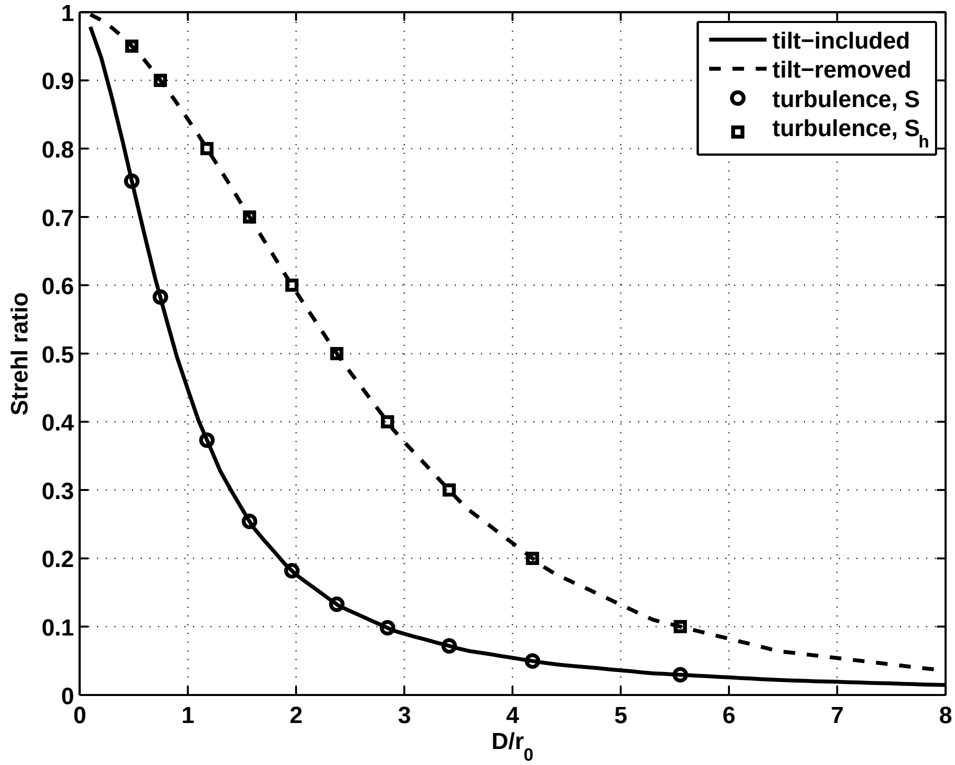

Figure 1: Tilt-included Strehl ratio (S) and tilt-removed Strehl ratio (S_h) with D/r_0.

2.3 (3) $\sigma_T(\lambda/D)$, Turbulence-Induced Tilt/Jitter

Column 3 lists the standard deviation of the angular wavefront tilt in units of (λ/D) due to the turbulence condition specified by D/r_0. The value of σ_T is computed as

$$\frac{\sigma_T}{\lambda/D} = \left[C_Z \frac{4}{\pi^2} \left(\frac{D}{r_0} \right)^{5/3} \right]^{1/2}, \tag{7}$$

where $C_Z = 0.4489$ is Noll's covariance matrix element [4] for the Zernike polynomial $N = 2, 3$. The wavefront tilt is a random variable with 0 mean which would cause a source to wander (or jitter) with a standard deviation σ_T.

2.4 (4) S_h, Tilt-Removed (Higher-Order) Strehl Ratio

Column 4 lists the tilt-removed Strehl ratio for a given turbulence condition, representing the on-axis intensity degradation due to higher-order turbulence only, i.e., as if the tilt whose standard deviation is listed in column 3 were completely removed from the random wavefront. To calculate the tilt-removed Strehl, the long-exposure OTF $\mathcal{H}_{LE}$ in Eq. (3) is replaced with the short-exposure OTF given by

$$\mathcal{H}_{SE}(\vec{f}) = \exp\left\{ -\frac{1}{2} 6.88 \left(\frac{|\bar{\lambda} d_i \vec{f}|}{r_o} \right)^{5/3} \left[1 - \left(\frac{|\bar{\lambda} d_i \vec{f}|}{D} \right)^{1/3} \right] \right\}. \tag{8}$$

The scaling parameter for this calculation is also D/r_0 as shown in Figure 1. The values indicated in column 4 of Table 1 are noted on the plot. These values were chosen arbitrarily to serve as a standard reference for the rest of the table.

3 Static Optical Relations

3.1 (5) S, Strehl Ratio

Column 5 lists the Strehl ratio values which are used for the equivalents in columns 6-8. These values are numerically the same as column 4, but pertain to the static optical relations in the columns to the right.

3.2 (6) $\sigma_J(\lambda/D)$, Strehl-Equivalent Jitter

Column 6 lists the jitter σ_j in units of (λ/D) which results in the Strehl ratio degradation in column 5 for a Gaussian fit with $\sigma = (\sqrt{2}/\pi)(\lambda/D) = 0.45(\lambda/D)$ to the diffraction-limited far-field irradiance for a uniform circular aperture [5]. This equivalence is derived from the relation

$$S_j = \frac{1}{1 + \frac{\pi^2}{2}[\sigma_j/(\lambda/D)]^2}, \tag{9}$$

and solving for σ_j as

$$\sigma_j/(\lambda/D) = \frac{\sqrt{2}}{\pi}\left(S_j^{-1} - 1\right)^{1/2}, \tag{10}$$

to yield the values listed in column 6.

3.3 (7) $\sigma_{WFE}(\lambda)$, Strehl-Equivalent Wavefront Error

Column 7 lists the rms aperture-averaged wavefront error (WFE) in units of (λ) which results in the Strehl ratio value listed in column 5. This equivalence is established by use of the Maréchal approximation [6] for which a commonly-used form is [5]

$$S \simeq \exp\left[-\left(\frac{2\pi}{\lambda}\sigma_{WFE}\right)^2\right]. \tag{11}$$

Taking this approximation as an equality and inverting the relation to solve for σ_{WFE} gives

$$\frac{\sigma_{WFE}}{\lambda} = \frac{[-\log(S)]^{1/2}}{2\pi}. \tag{12}$$

to yield the values listed in column 7.

3.4 (8) BQ, Strehl-Equivalent Beam Quality

Column 8 lists the approximate beam quality (BQ) which is equivalent to the Strehl value in column 5. This relation is derived by considering a Gaussian beam for which the aberrated dimension (radius, "sigma") is a and the diffraction-limited dimension is a_0. We define beam quality as the ratio of the aberrated beam dimension to the diffraction limit, i.e.,

$$BQ \equiv \frac{a}{a_0}. \tag{13}$$

The peak intensity of such a Gaussian beam is proportional to a^{-2} in the aberrated case, and a_0^{-2} in the diffraction-limited case. Thus, for the same total power in the Gaussian beam,

$$S \equiv \frac{a^{-2}}{a_0^{-2}} = \left(\frac{a}{a_0}\right)^{-2} = BQ^{-2}. \tag{14}$$

Thus, solving for BQ in terms of S:

$$BQ = S^{-1/2}, \tag{15}$$

which are the values listed in column 8 given S in column 5.

4 Code Listing for Generating Table

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Script to generate look-up table information for the back of the MZA
% business card.  Table gives relations for turbulence-induced Strehl and
% jitter as well as equivalent WFE and jitter for common optical
% aberrations.  Makes use of functions in ATMTools toolbox for MATLAB.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (c) 2012 MZA Associates Corporation, Dayton, Ohio
%  Author:  Matthew R. Whiteley, Ph.D.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
close all; clear variables;

% Generate LUTs for tilt-included and tilt-removed Strehl with D/r0
D = 1; % aperture diameter when needed
Dr0_LUT = [0.1:0.1:20];
[Strehl_LUT,StrehlTR_LUT] = OpenLoopStrehl(D./Dr0_LUT,D,0);

% turbulence values
Sh_Turb = [0.95 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1]'; % values in the LUt
Dr0 = interp1(StrehlTR_LUT,Dr0_LUT,Sh_Turb); % D/r0 for tilt-removed Strehl
S_Turb = interp1(Dr0_LUT,Strehl_LUT,Dr0); % tilt-included turbulence Strehl
sigJ_Turb = sqrt(NollMatrix(2)*(4/pi^2)*((Dr0).^(5/3))); % derived from Noll Z-tilt variance

% static WFE values
sigWFE = sqrt(-log(Sh_Turb))/(2*pi); % inverse Marechal approx
BQ = 1./sqrt(Sh_Turb); % from Sh = A_abb/A_dl
sigJ = (sqrt(2)/pi)*sqrt((Sh_Turb.^-1)-1); % from jitter Strehl = 1/(1+(pi^2/2)sigJ^2)

% Columns of the look-up table
LUT = [Dr0 S_Turb sigJ_Turb Sh_Turb sigJ sigWFE BQ]

figure
plot(Dr0_LUT,Strehl_LUT,'k-',Dr0_LUT,StrehlTR_LUT,'k--');
hold on
plot(Dr0,S_Turb,'ko',Dr0,Sh_Turb,'ks')
hold off
xlim([0 8])
xlabel('D/r_{0}')
ylabel('Strehl ratio')
legend('tilt-included','tilt-removed','turbulence, S','turbulence, S_{h}')

% cell2Excel(num2cell(LUT)); % to send to Excel
```

References

[1] D. L. Fried, "Optical resolution through a randomly inhomogeneous medium for very long and very short exposures," *J. Opt. Soc. Am.*, vol. 56, pp. 1372–1379, October 1966.

[2] R. J. Sasiela, *Electromagnetic Wave Propagation in Turbulence*. Berlin: Springer-Verlag, 1994.

[3] M. C. Roggemann and B. Welsh, *Imaging Through Turbulence*. Boca Raton: CRC Press, 1996.

[4] R. J. Noll, "Zernike polynomials and atmospheric turbulence," *J. Opt. Soc. Am.*, vol. 66, pp. 207–211, March 1976.

[5] P. Merritt, *Beam Control for Laser Systems*. Albuquerque: The Directed Energy Professional Society, 2012.

[6] M. Born and E. Wolf, *Principles of Optics*. Oxford: Pergamon Press, sixth ed., 1980.